# A Short Introduction to Lebesgue Integration

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This material is designed for a 3-hour lecture and contains only a brief overview of Lebesgue integration.
It is not suitable as a self-study resource for those encountering Lebesgue integration for the first time.

# Contents

# Invitation

## The Space of Continuous Functions

Let $C[a, b]$ be the collection of continuous real functions defined on the interval $[a, b]$, and let $p$ be a real number satisfying $1 \leq p < \infty$.

The metric $d_p$ on $C[a, b]$ is defined by

$$d_p(f, g) = \left( \int_a^b |f(x) - g(x)|^p \, dx \right)^{1/p}.$$

Unfortunately, the metric space $(C[a, b], d_p)$ is not complete.

What space is the completion of $(C[a, b], d_p)$?

# Invitation

## Length of an Interval and Its Generalization

Consider the subsets $A$, $B$ of $\mathbb{R}$.

Let $\ell(A)$ and $\ell(B)$ be the lengths of sets $A$ and $B$ (whenever it is defined).

It is quite natural to expect that the function $\ell$ would have the following properties:

- The value of length is non-negative.
- The length of $\varnothing$ is defined. (and equals zero.)
- If the lengths of $A$ and $B$ are defined, then the lengths of $A \cup B$, $A \cap B$, $A \setminus B$ are defined.
- If $A \subseteq B$, then $\ell(A) \leq \ell(B)$.
- If the length of $A \cap B$ is zero, then $\ell(A \cup B) = \ell(A) + \ell(B)$.

# Measure

### Definition: σ-Algebra.

A **σ-algebra** (also known as a **σ-field**) is a class $\Sigma$ of subsets of a set $X$ with the properties:

(a) Both $\varnothing$ and $X$ belong to $\Sigma$.

(b) $S \in \Sigma \Rightarrow X \setminus S \in \Sigma$.

(c) If $S_n \in \Sigma$ for $n = 1, 2, 3, \cdots$, then $\bigcup_{n=1}^{\infty} S_n \in \Sigma$.

A set $S \in \Sigma$ is said to be **measurable**.

**Corollary.** If $S_n \in \Sigma$ for $n = 1, 2, 3, \cdots$, then $\bigcap_{n=1}^{\infty} S_n \in \Sigma$.

# Measure

**Definition: Measure, Measure Space.**

Let $X$ be a set and let $\Sigma$ be a σ-algebra of subsets of $X$.

A function $\mu : \Sigma \to \overline{\mathbb{R}}^{+}$ is a **measure** if it has the properties:

(a) $\mu(\varnothing) = 0$; $\overline{\mathbb{R}}^{+} = [0, \infty]$

(b) $\mu$ is countably additive, that is, if $S_j \in \Sigma$ (for $j = 1, 2, 3, \cdots$) are pairwise disjoint sets, then

$$\mu\left(\bigcup_{j=1}^{\infty} S_j\right) = \sum_{j=1}^{\infty} \mu(S_j).$$

The triple $(X, \Sigma, \mu)$ is called a **measure space**.

If $\mu$ is a measure on $X$ and $\mu(X) < \infty$, then $\mu$ is called a **finite measure**.

In this case, $(X, \Sigma, \mu)$ is called a **finite measure space**.

# Measure

### Definition: "Almost Everywhere"

Let $(X, \Sigma, \mu)$ be a measure space.

- A set $S \in \Sigma$ with $\mu(S) = 0$ is said to have **measure zero** (or, is a **null set**).
- A given property $P(x)$ of points $x \in X$ is said to hold **almost everywhere** if the set

  $$\{\, x \mid P(x) \text{ is false} \,\}$$

  has measure zero; alternatively, the property $P$ is said to hold for **almost every** $x \in X$. The abbreviation "**a.e.**" will denote either of these terms.

# Measure

### Example 1. (Counting Measure)

Let $X = \mathbb{N}$, let $\Sigma_c$ be the class of all subsets of $\mathbb{N}$.

For any $S \subseteq \mathbb{N}$, define $\mu_c(S)$ to be the number of elements of $S$.

Then $\Sigma_c$ is a σ-algebra and $\mu_c$ is a measure on $\Sigma_c$.

This measure is called **counting measure** on $\mathbb{N}$.

✓ The only set of measure zero in this measure space is the empty set.

# Measure

**Example 2. (Lebesgue Measure)**

There is a σ-algebra $\Sigma_L$ in $\mathbb{R}$ and a measure $\mu_L$ on $\Sigma_L$ such that for any finite interval $I = [a, b]$,

$$I \in \Sigma_L \quad \text{and} \quad \mu_L(I) = \ell(I).$$

The sets of measure zero in this space are exactly those sets $A$ with the following property: for any $\epsilon > 0$ there exists a sequence of intervals $I_j \subseteq \mathbb{R}$, $j = 1, 2, 3, \cdots$, such that

$$A \subseteq \bigcup_{j=1}^{\infty} I_j \quad \text{and} \quad \sum_{j=1}^{\infty} \ell(I_j) < \epsilon.$$

This measure is called **Lebesgue measure** and the sets in $\Sigma_L$ are said to be **Lebesgue measurable**.

✓ Lebesgue measure is completely characterized by the above two properties.

## Measure

### Example 2. (Lebesgue Measure) (continued)

- Any countable subset of $\mathbb{R}$ has Lebesgue measure zero.
- The Cantor's ternary set is uncountable set but it has Lebesgue measure zero.
- If a set includes an interval of positive length, then the set has a positive measure.

There is no function $\mu$ from $\wp(\mathbb{R})$ to $[0, \infty]$ that satisfies all the following properties:

(i) If $I$ is an interval, then $\mu(I)$ is the length of $I$.

(ii) If $A$ has Lebesgue measure zero, then $\mu(A) = 0$.

(iii) If $\{A_j \mid j \in J\}$ is a collection of pairwise disjoint sets and $A = \bigcup_{j \in J} A_j$, then $\mu(A) = \sum_{j \in J} \mu(A_j)$.

# Measure

### Example 3. (Borel Measure)

Suppose that the usual topology is given on $\mathbb{R}$.

Then there exists the smallest σ-algebra $B(X)$ that contains all the open intervals of $\mathbb{R}$. An element of $B(X)$ is called a **Borel set**, and any measure defined on the σ-algebra of Borel sets is called a **Borel measure**.

# Measure

## Lebesgue Measure and Borel Measure

✓ Any Borel set is Lebesgue measurable.

✓ A Lebesgue measurable set is not always Borel measurable.

✓ If $\lambda$ is the Borel measure with the property $\lambda(I) = \ell(I)$ for any interval $I$, then the Lebesgue measure is the completion of $\lambda$. This idea extends to finite-dimensional spaces $\mathbb{R}^n$ but does not hold, in general, for infinite-dimensional spaces. Infinite-dimensional Lebesgue measures do not exist.

✓ Assignment. What is the **completion** of a measure?

# Lebesgue Integral

## The Steps of Defining the Lebesgue Integral

1. Simple Functions
2. Non-negative Simple Functions
3. Non-negative Measurable Functions
4. Measurable Functions
5. Complex-Valued Measurable Functions

# Lebesgue Integral

**Definition: Simple Functions.**

A function $\phi : X \to \mathbb{R}$ is **simple** if it has the form

$$\phi = \sum_{j=1}^{k} \alpha_j \chi_{S_j},$$

for some $k \in \mathbb{N}$, where $\alpha_j \in \mathbb{R}$ and $S_j \in \Sigma$, $j = 1, 2, 3, \cdots, k$.

In this definition, $S_j$'s are not necessarily pairwise disjoint. But any simple function is expressible in the form of the sum of simple functions where $S_j$'s are pairwise disjoint and all $\alpha_j$'s are distinct.

# Lebesgue Integral

**Definition: Lebesgue Integral of a Non-negative Simple Function.**

If $\phi$ is non-negative and simple, then the **integral** of $\phi$ (over $X$, with respect to $\mu$) is defined to be

$$\int_X \phi \, d\mu = \sum_{j=1}^{k} \alpha_j \mu(S_j).$$

(We allow $\mu(S_j) = \infty$ here, and we use the algebraic rules in $\overline{\mathbb{R}}^+$.)
The value of the integral can be infinity.

# Lebesgue Integral

**Example 4.** (Practice)

Let $S_1 = [1,\ 3]$, $S_2 = (2,\ 5]$, $S_3 = (0,\ 4]$, $X = S_1 \cup S_2 \cup S_3$, and let

$$\phi = 3\chi_{S_1} + 5\chi_{S_2} + 7\chi_{S_3}.$$

(1) Find the decomposition for $\phi$, that is, express $\phi$ in the (canonical) form

$$\phi = \sum_{j=1}^{k} \alpha_j \chi_{A_j}$$

where $A_j$'s are pairwise disjoint and $\alpha_j$'s are distinct.

(2) Find $\int_X \phi \, d\mu$ where $\mu$ is the Lebesgue measure.

# Lebesgue Integral

**Example 5.** (Practice)

Find the integral

$$\int_{[0,\ 1]} f \, d\mu$$

where $f(x) = 1$ if $x$ is rational, and $f(0) = 0$ otherwise, and $\mu$ is the Lebesgue measure.

# Lebesgue Integral

**Definition: Lebesgue Integral of a Non-negative Measurable Function.**

A function $f : X \to \overline{\mathbb{R}}$ is said to be **measurable** if, for every $\alpha \in \mathbb{R}$,

$$\boxed{f^{-1}((\alpha, \infty])} \in \Sigma.$$

$f^{-1}((\alpha, \infty]) = \{x \in X \mid f(x) > \alpha\}$

If $f$ is measurable then the function $|f| : X \to \overline{\mathbb{R}}$ and $f^{\pm} : X \to \overline{\mathbb{R}}$, defined by

$$|f|(x) = |f(x)|, \quad f^{\pm}(x) = \max\{\pm f(x), 0\}$$

are measurable.

If $f$ is non-negative and measurable, then the integral of $f$ is defined to be

$$\int_X f \, d\mu = \sup \left\{ \int_X \phi \, d\mu \,\middle|\, \phi \text{ is simple and } 0 \leq \phi \leq f \right\}.$$

# Lebesgue Integral

### Theorem: Measurability of a Function.

A function $f : X \to \overline{\mathbb{R}}$ is measurable if and only if one of the following is true.

(a) For every $\alpha \in \mathbb{R}$, $f^{-1}([-\infty, \alpha)) \in \Sigma$.

(b) For every measurable subset $S$ of $\mathbb{R}$, $f^{-1}(S) \in \Sigma$.

(c) For every open subset $S$ of $\mathbb{R}$, $f^{-1}(S) \in \Sigma$.

### Theorem: Algebra of Measurable Functions.

(a) If $f : X \to \overline{\mathbb{R}}$ and $g : X \to \overline{\mathbb{R}}$ are measurable, then $f+g$, $f-g$, $fg$ are measurable.
If $g(x) \neq 0$ for all $x$, then $f/g$ is measurable.

(b) $f : X \to \overline{\mathbb{R}}$ is measurable if and only if both $f^{+}$ and $f^{-}$ are measurable.

(c) If $\{f_n\}$ is a sequence of measurable functions,
then $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, $\liminf_n f_n$ are measurable.

# Lebesgue Integral

**Example 6.** (Practice)

Find the integral

$$\int_{[0,\ 3]} f\, d\mu$$

where $f(x) = x^2$ and $\mu$ is the Lebesgue measure.

What if $f$ is any continuous function?

# Lebesgue Integral

**Definition: Lebesgue Integral of a Measurable Function.**

If $f$ is measurable and $\int_X |f| \, d\mu < \infty$, then $f$ is said to be **integrable** .

If $f$ is integrable, then the **integral** of $f$ is defined to be

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu.$$

It can be shown that if $f$ is integrable then each of the terms on the right of this definition are finite, so there is no problem with a difference such as $\infty - \infty$ .

# Lebesgue Integral

**Definition: Lebesgue Integral of a Complex-Valued Function.**

A complex-valued function $f$ is said to be **integrable** if the real and imaginary parts of $f$ are integrable, and the the **integral** of $f$ is defined to be

$$\int_X f\,d\mu = \int_X \mathrm{Re}\, f\,d\mu + \boldsymbol{i}\int_X \mathrm{Im}\, f\,d\mu.$$

# Lebesgue Integral

## Definition: Lebesgue Integral of a Function on a Subset.

Suppose that $S \in \Sigma$ and $f$ is a real or complex-valued function on $S$.

Extend $f$ to a function $\tilde{f}$ on $X$ by defining $\tilde{f}(x) = 0$ for $x \notin S$.

Then $f$ is said to be **integrable** (over $S$) if $\tilde{f}$ is integrable (over $X$), and we define

$$\int_S f \, d\mu = \int_X \tilde{f} \, d\mu.$$

The set of $\mathbb{F}$-valued integrable functions on $X$ will be denoted by $\mathscr{L}^1_{\mathbb{F}}(X)$.

# Lebesgue Integral

**Example 7. Integral with Respect to the Counting Measure.**

Suppose that $(X, \Sigma, \mu) = (\mathbb{N}, \Sigma_c, \mu_c)$. Any function $f : \mathbb{N} \to \mathbb{F}$ can be regarded as an $\mathbb{F}$-valued sequence $\{a_n\}$ with $a_n = f(n)$, and since all subsets of $\mathbb{N}$ are measurable, every such sequence $\{a_n\}$ can be regarded as a measurable function.

A sequence $\{a_n\}$ is integrable (with respect to $\mu_c$) if and only if $\sum_{n=1}^{\infty} |a_n| < \infty$;

The integral of $\{a_n\}$ is simple the sum $\sum_{n=1}^{\infty} a_n$.

Instead of the general notation $\mathscr{L}^1(\mathbb{N})$, the space of such sequences will be denoted by $\ell^1$ or $\ell^1_{\mathbb{F}}$.

# Lebesgue Integral

**Definition: Lebesgue Integrable Functions.**

Let $(X, \Sigma, \mu) = \left(\mathbb{R}^k, \Sigma_L, \mu_L\right)$, for some $k \geq 1$. If $f \in \mathscr{L}^1\left(\mathbb{R}^k\right)$ (or $f \in \mathscr{L}^1(S)$ with $S \in \Sigma_L$), then $f$ is said to be **Lebesgue integrable**.

# Properties of Lebesgue Integral

**Theorem: Relation between Riemann and Lebesgue Integrals.**

Let $I = [a, b] \subseteq \mathbb{R}$ is a bounded interval, and $f : I \to \mathbb{R}$ is a bounded function.

If $f$ is Riemann integrable on $I$, then $f$ is Lebesgue integrable on $I$.

In this case, the values of the two integrals of $f$ coincide.

# Properties of Lebesgue Integral

**Theorem: Elementary Properties.**

Let $(X, \Sigma, \mu)$ be a measure space and let $f \in \mathscr{L}^1(X)$.

(a) If $f(x) = 0$ a.e., then $f \in \mathscr{L}^1(X)$ and $\int_X f\, d\mu = 0$.

(b) If $\alpha \in \mathbb{R}$ and $f, g \in \mathscr{L}^1(X)$, then the functions $f + g$ and $\alpha f$ belong to $\mathscr{L}^1(X)$, and

$$\int_I (f+g) d\mu = \int_X f\, d\mu + \int_X g\, d\mu, \quad \int_I \alpha f\, d\mu = \alpha \int_X f\, d\mu.$$

In particular, $\mathscr{L}^1(X)$ is a vector space.

(c) If $f, g \in \mathscr{L}^1(X)$ and $f(x) \le g(x)$ for a.e. $x \in X$, then $\int_X f\, d\mu \le \int_X g\, d\mu$.

If $f(x) < g(x)$ for a.e. $x \in S$, with $\mu(S) > 0$, then $\int_S f\, d\mu < \int_S g\, d\mu$.

# Properties of Lebesgue Integral

(continued)

(d) An integrable function is a.e. finite.

(e) If $f$ is a measurable function and $A$ is a measurable set, then

$$\mu(A)\inf f \leq \int_A f \, d\mu \leq \mu(A)\sup f .$$

(f)
$$\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu .$$

**Theorem: Lebesgue Integral Induces a Measure.**

If $f$ is measurable with $f \geq 0$, then $A \mapsto \int_A f \, d\mu$ is a measure on $X$.

(What is the converse of this theorem?)

# Properties of Lebesgue Integral

**Definition: Essential Supremum.**

Suppose that $f$ is a measurable function and there exists a number $b$ such that $f(x) \leq b$ a.e. Then we can define the **essential supremum** of $f$ to be

$$\operatorname{esssup} f = \inf \{b \mid f(x) \leq b \text{ a.e.}\}.$$

The **essential infimum** of $f$ can be defined similarly.

- ✓ It is a simple (but not completely trivial) consequence of this definition that $f(x) \leq \operatorname{esssup} f$ a.e.
- ✓ A measurable function $f$ is said to be **essentially bounded** if there exists a number $b$ such that $|f(x)| \leq b$ a.e.

# Properties of Lebesgue Integral

## Definition: $L^1$ Space.

If we define $d_1(f, g)$ for $f, g \in \mathscr{L}^1(X)$ by

$$d_1(f, g) = \int_X |f - g| \, d\mu,$$

$d_1$ is not a metric since $d_1(f, g) = 0$ does not imply $f = g$.

Consider an equivalent relation on $\mathscr{L}^1(X)$ defined by

$$f \sim g \quad \text{iff} \quad f(x) = g(x) \text{ for a.e. } x \in X,$$

and denote the quotient space $\mathscr{L}^1(X)/\sim$ by $L^1(X)$.

$L^1(X)$ is a vector space, with the definitions

$$[f] + [g] = [f + g], \quad \alpha[f] = [\alpha f], \quad 0 = [0].$$

# Properties of Lebesgue Integral

(continued)

For $[f], [g] \in L^1(X)$, define

$$d_1([f], [g]) = \int_X |f - g| \, d\mu,$$

then $d_1$ is a metric on $L^1(X)$.

Instead of writing $[f] \in L^1(X)$, we just write $f \in L^1(X)$.

# $L^p$ Space

**Definition: $\mathrm{L}^p$ Space.**

Define the sets

$$\mathscr{L}^p(X) = \left\{ f \;\middle|\; f \text{ is measurable and } \left( \int_X |f|^p \, d\mu \right)^{1/p} < \infty \right\}, \; 1 \leq p < \infty.$$

$$\mathscr{L}^\infty(X) = \{ f \mid f \text{ is measurable and } \operatorname{esssup} |f| < \infty \}.$$

We also define the corresponding sets $L^p(X)$.

When $X$ is a bounded interval $[a, b]$ and $1 \leq p \leq \infty$, we write $L^p[a, b]$.

# $L^p$ Space

**Theorem: Minkowski's and Hölder's Inequalities.**

Suppose that $f$ and $g$ are measurable functions. Then the following inequalities hold. (Infinite values are allowed.)

- **Minkowski's inequality** (for $1 \leq p < \infty$)

$$\left(\int_X |f+g|^p \, d\mu\right)^{1/p} \leq \left(\int_X |f|^p \, d\mu\right)^{1/p} + \left(\int_X |g|^p \, d\mu\right)^{1/p},$$

$$\operatorname{esssup}|f+g| \leq \operatorname{esssup}|f| + \operatorname{esssup}|g|.$$

- **Hölder's inequality** (for $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$)

$$\int_X |fg| \, d\mu \leq \left(\int_X |f|^p \, d\mu\right)^{1/p} \left(\int_X |g|^q \, d\mu\right)^{1/q},$$

$$\int_X |fg| \, d\mu \leq \operatorname{esssup}|f| \int_X |g| \, d\mu.$$

## $L^p$ Space

Since the previous inequalities hold in the sense of Riemann integral, it is worth taking a look at the proofs. Fix $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$.

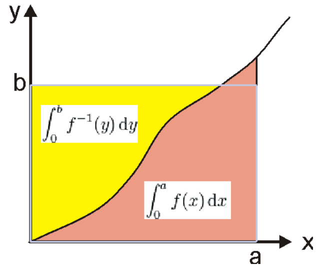
**Step 1. Young's inequality.**

If $f(x) = x^{p-1}$, then $f^{-1}(y) = y^{q-1}$.

The are of the rectangle $a$, $b$ cannot be larger than sum of the areas under the functions $f$ and $f^{-1}$, that is

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy,$$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$



# $L^p$ Space

**Step 2. Hölder's Inequality**

It is sufficient to prove for the case $0 < \|f\|_p < \infty$ and $0 < \|g\|_q < \infty$.

Let $F = f/\|f\|_p$, $G = g/\|g\|_q$, then $\|F\|_p = \|G\|_q = 1$.

By Young's inequality,

$$|F(s)G(s)| \leq \frac{|F(s)|^p}{p} + \frac{|G(s)|^q}{q}.$$

Integrating both sides gives

$$\|FG\|_1 \leq \frac{\|F\|_p^p}{p} + \frac{\|G\|_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1,$$

which proves the theorem.

**Step 3. Minkowski's Inequality**

Since $h(x) = |x|^p$ is a convex function for $x > 0$,

$$\left|\frac{1}{2}f + \frac{1}{2}g\right|^p \leq \left|\frac{1}{2}|f| + \frac{1}{2}|g|\right|^p \leq \frac{1}{2}|f|^p + \frac{1}{2}|g|^p.$$

This means that

$$|f+g|^p \leq \frac{1}{2}|2f|^p + \frac{1}{2}|2g|^p = 2^{p-1}|f|^p + 2^{p-1}|g|^p.$$

# $L^p$ Space

(Continued) Now, by Hölder's inequality, we find that

$$\begin{aligned}
\|f+g\|_p^p &= \int |f+g|^p \, d\mu = \int |f+g| \cdot |f+g|^{p-1} \, d\mu \\
&\leq \int (|f|+|g|) |f+g|^{p-1} \, d\mu \\
&= \int |f| \, |f+g|^{p-1} \, d\mu + \int |g| \, |f+g|^{p-1} \, d\mu \\
&\leq \left( \left( \int |f|^p \, d\mu \right)^{1/p} + \left( \int |g|^p \, d\mu \right)^{1/p} \right) \left( \int |f+g|^{(p-1)\left(\frac{p}{p-1}\right)} \, d\mu \right)^{1-\frac{1}{p}} \\
&= (\|f\|_p + \|g\|_p) \frac{\|f+g\|_p^p}{\|f+g\|_p}.
\end{aligned}$$

## $L^p$ Space

**Corollary.** Suppose that $1 \leq p \leq \infty$.

(a) $L^p(X)$ is a vector space.

(b) The function $d_p(f, g)$ defined by

$$d_p(f, g) = \left( \int_X |f - g|^p \, d\mu \right)^{1/p} \quad \text{if } 1 \leq p < \infty,$$
$$d_p(f, g) = \operatorname{esssup} |f - g| \quad \text{if } p = \infty$$

is a metric on $L^p(X)$. This metric will be called the **standard metric** on $L^p(X)$. Unless otherwise stated, $L^p(X)$ will be assumed to have this metric.

## Example 8: $L^p$ Space with Counting Measure.

In the special case where $(X, \Sigma, \mu) = (\mathbb{N}, \Sigma_c, \mu_c)$, the space $L^p(\mathbb{N})$ consists of the set of sequences $\{a_n\}$ in $\mathbb{F}$ with the property that

$$\left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} < \infty \qquad \text{for } 1 \le p < \infty,$$

$$\sup\{|a_n| \mid n \in \mathbb{N}\} < \infty \qquad \text{for } p = \infty.$$

These spaces will be denoted by $\ell^p$ (or $\ell^p_{\mathbb{F}}$).

Since there are no sets of measure zero in this measure space, there is no question of taking equivalent classes here. The spaces $\ell^p$ are both vector spaces and metric spaces.

# $L^p$ Space

By using counting measure and letting $x$ and $y$ be sequences in $\mathbb{F}$ (or elements of $\mathbb{F}^k$ for some $k \in \mathbb{N}$), we can obtain the following important special case of the previous theorem.

**Corollary: Minkowski's and Hölder's Inequalities.**

- **Minkowski's inequality** (for $1 \leq p < \infty$)

$$\left( \sum_{j=1}^{k} |x_j + y_j|^p \right)^{1/p} \leq \left( \sum_{j=1}^{k} |x_j|^p \right)^{1/p} + \left( \sum_{j=1}^{k} |y_j|^p \right)^{1/p}.$$

- **Hölder's inequality** (for $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$)

$$\sum_{j=1}^{k} |x_j y_j| \leq \left( \sum_{j=1}^{k} |x_k|^p \right)^{1/p} \left( \sum_{j=1}^{k} |y_j|^q \right)^{1/q}.$$

Here, $k$ and the values of the sums may be infinity.

# $L^p$ Space

**Corollary: Cauchy–Bunyakovsky–Schwarz Inequality.**

$$\sum_{j=1}^{k} |x_j|\,|y_j| \le \left(\sum_{j=1}^{k} |x_j|^2\right)^{1/2} \left(\sum_{j=1}^{k} |y_j|^2\right)^{1/2}.$$

# $L^p$ Space

**Theorem: Completeness of $L^p$.**

Suppose that $1 \leq p \leq \infty$. Then the metric space $L^p(X)$ is complete.
In particular, the sequence space $\ell^p$ is complete.

**Theorem: Completion of $C[a, b]$.**

Suppose that $[a, b]$ is a bounded interval and $1 \leq p < \infty$.
Then the set $C[a, b]$ is dense in $L^p[a, b]$.

# A Few Fundamental Theorems

**Fatou's Lemma.**

If $\{f_n\}$ is a sequence of non-negative measurable functions and

$$f(x) = \liminf_{n \to \infty} f_n(x) \text{ a.e. } x \in X,$$

then $f$ is measurable and

$$\int_X f \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu.$$

# A Few Fundamental Theorems

**Monotone Convergence Theorem.**

If $\{f_n\}$ is a sequence of non-negative measurable functions, and $\{f_n\}$ increases monotonically to $f(x)$ for each $x$, that is, $f_n \nearrow f$ pointwise, then $f$ is measurable and

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

**Corollary.**

Suppse that $\{f_n\}$ and $f$ are non-negative and measurable. If $\{f_n\}$ increases to $f$ almost everywhere, then

$$\int_X f_n \, d\mu \nearrow \int_X f \, d\mu.$$

## A Few Fundamental Theorems

### Example 10. (A Counterexample in Riemann Integral)

Let $\{r_n\}$ be a sequence that is one to one function from $\mathbb{N}$ onto $\mathbb{Q} \cap [0,\ 2]$.

Define $f_n : [0,\ 2] \rightarrow \mathbb{R}$ by $f_n(x) = 1$ if $x \in \{r_1,\ r_2,\ \cdots,\ r_n\}$ and $f_n(x) = 0$ otherwise.

Then $f_n(x)$ converges to the characteristic function $\chi_{\mathbb{Q}}(x)$ as $n \rightarrow \infty$, for each $x$.

For each $n$, $f_n$ is Riemann integrable on $[0,\ 2]$ for each $n$, but the limit function $\chi_{\mathbb{Q}}$ is not Riemann integrable.

Therefore ‘Monotone Convergence Theorem’ is not valid for Riemann integrals.

# A Few Fundamental Theorems

## The Dominated Convergence Theorem (Version 1)

Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ on $X$, where $g$ is integrable on $X$. (In this case, we say "$f_n$ is **dominated** by $g$.")

If

$$f(x) = \lim_{n \to \infty} f_n(x) \text{ a.e. } x \in X \text{ pointwise,}$$

then $f$ is integrable on $X$ and

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

# A Few Fundamental Theorems

## The Dominated Convergence Theorem (Version 2)

Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ **a.e.** on $X$, where $g$ is integrable on $X$. Suppose that $X$ is **complete**.
($X$ is said to be **complete** if every subset of any null set is measurable.)

If

$$f(x) = \lim_{n \to \infty} f_n(x) \text{ a.e. } x \in X \text{ pointwise,}$$

then $f$ is integrable on $X$ and

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

# A Few Fundamental Theorems

## Example 11. (An Application of the Dominated Convergence Theorem)

Suppose that

- $\{f_n\}$ is a sequence of Riemann integrable functions (on $[a, b]$),
- $g$ is a Riemann integrable function (on $[a, b]$),
- $|f_n(x)| \leq g(x)$ for each $n$ and $x$,
- The limit function $f$ of $\{f_n\}$ is Riemann integrable.

Show that

$$\lim_{n \to \infty} \int_a^b f_n(x)\,dx = \int_a^b \left( \lim_{n \to \infty} f_n(x) \right) dx,$$

where the integrals represent the Riemann integrals.

# A Few Fundamental Theorems

**Corollary: Bounded Convergence Theorem.**

Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq M$ on $X$, for some positive number $M$. (In this case, we say "$f_n$ is **uniformly bounded** on $X$.")

If

$$f(x) = \lim_{n \to \infty} f_n(x) \text{ a.e. } x \in X \text{ pointwise,}$$

then $f$ is integrable on $X$ and

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

# A Few Fundamental Theorems

**Theorem: Relation between Improper Riemenn Integral and Lebesgue Integral.**

If the improper Riemann integral

$$\int_{-\infty}^{\infty} f(x)dx$$

converges absolutely (with no singularity inside), then the Lebesgue integral

$$\int_{\mathbb{R}} f \, d\mu$$

exists and equals the improper Riemann integral.

**Problem.** Consider the case of an integral of an unbounded function with a singularity at the endpoint of a bounded interval. How does this compare to the previous theorem?

# Radon-Nikodym Theorem

### Definition: Absolute Continuity.

Let $\Sigma$ be a σ-algebra, and let $\mu$ and $\nu$ be two measures defined on $\Sigma$. If $\nu(A)=0$ for every set $A$ for which $\mu(A)=0$, then $\nu$ is said to be **absolutely continuous with respect to** $\mu$.

This is written as $\nu \ll \mu$.

# Radon-Nikodym Theorem

**Radon-Nikodym Theorem.**

Let $\nu(X) < \infty$ and $\mu(X) < \infty$. If $\nu$ is absolutely continuous with respect to $\mu$, then there exists a $\Sigma$-measurable function $g : X \to [0, \infty)$ such that

$$\nu(A) = \int_A g \, d\mu \text{ for every } \mu\text{-measurable set } A. \qquad (*)$$

In this case, $g$ is called **Radon-Nikodym derivative** with respective to $\mu$, and denoted by

$$g = \frac{d\nu}{d\mu}.$$

Using this notation, (*) can be rewritten as:

$$\nu(A) = \int_A \frac{d\nu}{d\mu} d\mu.$$

# Radon-Nikodym Theorem

### Example 12. (Radon-Nikodym Derivative)

Let $X=[1,\ 7]$ and $\Sigma_L$ be the collection of Lebesgue measurable subsets of $X$. Suppose that $\mu$ be the Lebesgue measure defined for $\Sigma_L$.

(a) Define a measure $\nu$ on $X$ by $\nu([a,\ b])=3(b-a)$ for closed intervals, and extend $\nu$ on $\Sigma_L$. Find $\dfrac{d\nu}{d\mu}$.

(b) Define a measure $\nu$ on $X$ by $\nu([a,\ b])=\dfrac{1}{2}(b^2-a^2)$ for closed intervals, and extend $\nu$ on $\Sigma_L$. Find $\dfrac{d\nu}{d\mu}$. (Show that $\nu$ is a measure.)

Give a geometric interpretation for the results of (a) and (b).

# Radon-Nikodym Theorem

## Applications in Probability Theory.

The theorem is very important in extending the ideas of probability theory from probability masses and probability densities defined over real numbers to probability measures defined over arbitrary sets. It tells if and how it is possible to change from one probability measure to another. Specifically, the probability density function of a random variable is the Radon–Nikodym derivative of the induced measure with respect to some base measure (usually the Lebesgue measure for continuous random variables).

For example, it can be used to prove the existence of conditional expectation for probability measures. The latter itself is a key concept in probability theory, as conditional probability is just a special case of it.

(Wikipedia: Radon–Nikodym theorem, Accessed on August 18, 2024.)

# Further Topics for Study

- Product Measure and Fubini's Theorem
- Derivatives of Integrals
- Weak Convergence
- Applications in Probability Theory

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