Section 7.1

The Logarithm Defined as an Integral

[Thomas' Calculus pp.434-442]

WHAT TO LEARN

- ✓ A Brief Observation of Several Definitions
- ✓ Definitions of Logarithms and Exponentials
- ✓ Formulas Involving Logarithms and Exponentials

Several Definitions

Definitions in High Schools.

Let $a > 0$.

(1) Take canonical definition of a^x for rational x.

(2) Define $a^x = \lim_{n\to\infty} a^{x_n}$ where $\{x_n\}$ is a rational sequence satisfying $x_n \to x$.

(3) Define $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

(4) Define $\log_a x$ as the inverse function of a^x.

(5) Define $\ln x = \log_e x$.

Several Definitions

Analytic Definitions.

Let $a > 0$.

(1) Define $\exp(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(2) Define $\ln x$ as the inverse function of $\exp x$.

(3) Define $e^x = \exp(x)$ and $a^x = e^{x \ln a}$.

(4) Define $\log_a x = \dfrac{\ln x}{\ln a}$.

DEFINITIONS

Natural Logarithm: $\ln x = \int_1^x \frac{1}{t}\, dt$ for $x > 0$.

Natural Exponential Function: $e^x = \exp(x) = \ln^{-1} x \quad (x \in \mathbb{R})$

Natural Constant: $e = \ln^{-1} 1$ i.e., $\ln e = 1$.

General Exponential Functions: $a^x = e^{x \ln a} \quad (a > 0,\ a \neq 1,\ x \in \mathbb{R})$

General Logarithm Functions: $\log_a x =$ (the inverse function of a^x)

Definition. $$\ln x = \int_1^x \frac{1}{t}\,dt \quad (x > 0,\ x \text{ real})$$

Formulas.

(1) $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$

(2) $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x} \quad (x \neq 0)$

(3) $\ln bx = \ln b + \ln x$

(4) $\ln \dfrac{b}{x} = \ln b - \ln x$

(5) $\ln \dfrac{1}{x} = -\ln x$

(6) $\ln x^r = r \ln x \quad (r \text{ rational})$

(7) $\lim\limits_{x\to\infty} \ln x = \infty \quad \left(\lim\limits_{x\to 0^+} \ln x = -\infty\right)$

(8) $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C \quad (x \neq 0)$

Properties and Formulas

Definition. $\exp(x) = \ln^{-1} x$ (x real), $e = \ln^{-1} 1$.

Properties.

(1) $\ln$ and $\exp$ are increasing functions.

(2) $\ln$ and $\exp$ are one-to-one functions.

(3) $e^r = \exp(r)$ for all the rationals r.

Definition. $e^x = \exp(x)$ (x real)

Formulas.

(1) $e^{\ln x} = x$ ($x > 0$, x real)

(2) $\ln(e^x) = x$ (x real)

(3) $\dfrac{d}{dx} e^x = e^x$

(4) $\displaystyle\int e^x \, dx = e^x + C$

(5) $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$

(6) $e^{-x} = \dfrac{1}{e^x}$

(7) $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$

(8) $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

(9) $\displaystyle\lim_{x \to -\infty} e^x = 0$

(10) $\displaystyle\lim_{x \to \infty} e^x = \infty$

(11) $\displaystyle\int e^x \, dx = e^x + C$

Definition. $a^x = e^{x \ln a} \quad (a > 0,\ x \text{ real})$

Formulas.

(1) $a^{x_1} \cdot a^{x_2} = a^{x_1 + x_2}$

(2) $a^{-x} = \dfrac{1}{a^x}$

(3) $\dfrac{a^{x_1}}{a^{x_2}} = a^{x_1 - x_2}$

(4) $(a^{x_1})^{x_2} = a^{x_1 x_2} = (a^{x_2})^{x_1}$

(5) $\dfrac{d}{dx} a^x = a^x \ln a$

(6) $\displaystyle\int a^x \, dx = \frac{a^x}{\ln a} + C \quad (a \neq 1)$

Properties and Formulas

Definition. $\log_a x = (\text{the inverse function of } a^x) \quad (a > 0,\ a \neq 1,\ x > 0)$

Formulas.

(1) $a^{\log_a x} = x \quad (x > 0)$

(2) $\log_a (a^x) = x \quad (x \in \mathbb{R})$

(3) $\log_a x = \dfrac{\ln x}{\ln a}$

(4) $\log_a xy = \log_a x + \log_a y$

(5) $\log_a \dfrac{x}{y} = \log_a x - \log_a y$

(6) $\log_a \dfrac{1}{y} = -\log_a y$

(7) $\log_a x^y = y \log_a x$

(8) $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$

WHAT WE LEARNED

- ✓ A Brief Observation of Several Definitions
- ✓ Definitions of Logarithms and Exponentials
- ✓ Formulas Involving Logarithms and Exponentials